Quantum Tomography Via Group Theory

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Amongst the multitude of state reconstruction techniques, the so–called "quantum tomography" seems to be the most fruitful. In this letter, I will start by developing the mathematical apparatus of quantum tomography and, later, I will explain how it can be applied to various quantum systems.

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Quantum tomography, like all state reconstruction methods, is concerned with the problem of measuring the density matrix ρ of a physical system. The first observations on this topic are dated 1957 [1], but the greatest advancements are a result of this decade's work. Also the first experiments were performed during this decade [2,3]. The word tomography stems from medicine, as a consequence of the resemblance between the main formula of the first quantum tomography method, *i.e.* homodyne tomography [4–6], and the inverse Radon transform used in the CAT [6,7]. Nowadays, however, quantum tomography does not have much in common with its medical ancestor. Quantum tomography is a general term, referring to any state measurement procedure descending from an equation of the same form of equation (9). It is a versatile and mighty technique, as it can be applied to a great variety of systems and as it includes other methods as special cases.

Homodyne tomography became a particular case of quantum tomography, when group representation theory was employed [8]. Even though the latter led to a breakthrough in quantum tomography, it was recognized not to be the most general approach. There were, in fact, tomographic formulas (formulas like (9)) that could not be ascribed to standard group representation theory. Hence, in order to include these cases in a general mathematical framework, I will introduce some conditions that comprise the definition of group representation. Let, then, $\mathcal{G}$ be a group, $\mathcal{H}$ a separable Hilbert space, $\mathcal{B}(\mathcal{H})$ the algebra of linear bounded operators defined in all $\mathcal{H}$, namely Banach algebra [9], and T a linear mapping from $\mathcal{G}$ to $\mathcal{B}(\mathcal{H})$ (from now on, the word linear, when referring to an operator, will always be understood). If we can find an irriducible, unitary ray representation D of $\mathcal{G}$ (a ray representation is such that $D(g)D(h) = \mathrm{e}^{\mathrm{i}\xi_{gh}} D(g \cdot h)$, with $\xi \in \mathbb{R}$ [10]) and six correspondences $\alpha, \beta : \mathcal{G} \times \mathcal{G} \to \mathbb{R}$ and $f_{ij} : \mathcal{G} \to \mathcal{G}$, with $i, j = 1, 2$, satisfying the equations

$$D(g)\, T(h) = \mathrm{e}^{\mathrm{i}\alpha_{gh}}\, T(f_{11}(g) \cdot h \cdot f_{12}(g))\,, \tag{1}$$

$$D(g)\, T^{\dagger}(h) = \mathrm{e}^{\mathrm{i}\beta_{gh}}\, T^{\dagger}(f_{21}(g) \cdot h \cdot f_{22}(g))\,, \tag{2}$$

for every $g, h \in \mathcal{G}$, then we will say that $\{T(g)\}$ is a tomographic set. If T is an irriducible, unitary ray representation then $\{T(g)\}$ is a tomographic set (choose $D = T, f_{11}(g) = g, \dots$). The converse is not true in general, so the requirement of T being a representation is more stringent. When $\mathcal{H}$ is finite–dimensional, the hypothesis that $\{T(g)\}$ is a tomographic set is sufficient to derive (9), however the case of $\dim(\mathcal{H}) = \infty$ needs a further condition to make sure that every expression converges and can be attributed a precise mathematical meaning. More explicitly, T needs to fulfill the following inequality [11] :

$$\sum_g \, |\langle u|T(g)|v\rangle|^2 < \infty \qquad \forall\, |u\rangle, |v\rangle \in \mathcal{H}. \tag{3}$$

Condition (3) could be quite a nuisance, since it must be checked for every couple of vectors $|u\rangle, |v\rangle \in \mathcal{H}$. Fortunately, (3) is equivalent to

$$\exists\, |u\rangle, |v\rangle \in \mathcal{H} \;:\; \sum_g \, |\langle u|T(g)|v\rangle|^2 < \infty, \tag{4}$$

as I will demonstrate now. Let us start by admitting the validity of (4) for two vectors $|w_1\rangle, |w_2\rangle$ (eventually coincident) and consider the set V of vectors $|v\rangle \in \mathcal{H}$ of the form

$$|v\rangle = \sum_g v_g\, D(g)\, |w_i\rangle, \quad \text{with } \sum_g |v_g| < \infty\,, \tag{5}$$

i being 1 or 2. V is a linear manifold of $\mathcal{H}$ because $\alpha|v_1\rangle + \beta|v_2\rangle$ belongs to V if $|v_1\rangle, |v_2\rangle \in V$ and $\alpha, \beta \in \mathbb{C}$, since $\sum_g |\alpha\, v_{1g} + \beta\, v_{2g}| \leqslant |\alpha| \sum_g |v_{1g}| + |\beta| \sum_g |v_{2g}| < \infty$. The application of any $D(h)$ to a vector $|v\rangle \in V$, yields a vector that still belongs to V, in fact $D(h)|v\rangle = \sum_g v_g\, \mathrm{e}^{\mathrm{i}\xi_{hg}}\, D(h \cdot g)\, |w_i\rangle = \sum_g v_{h^{-1} \cdot g}\, \mathrm{e}^{\mathrm{i}\xi_{h(h^{-1} \cdot g)}}\, D(g)\, |w_i\rangle$, with $\sum_g |v_{h^{-1} \cdot g}\, \mathrm{e}^{\mathrm{i}\xi_{h(h^{-1} \cdot g)}}| = \sum_g |v_g| < \infty$. The irriducibility of the operators $D(g)$ implies that $V = \mathcal{H}$ or, in other words, that every vector in $\mathcal{H}$ can be written in the form (5). Hence for every $|u\rangle, |v\rangle \in \mathcal{H}$, with the help of Cauchy's inequality, we obtain

$$\begin{aligned} &\sum_g \, |\langle u|T(g)|v\rangle|^2 = \\ &= \sum_g \left| \sum_{g_1, g_2} u^*_{g_1}\, v_{g_1}\, \langle w_1|D^{\dagger}(g_1)\, T^{\dagger}(g)\, D(g_2)|w_2\rangle \right|^2 \leqslant \\ &\leqslant \left(\sum_{g_1} |u_{g_1}| \right)^2 \left(\sum_{g_2} |v_{g_2}| \right)^2 \sum_g \, |\langle w_1|T(g)|w_2\rangle|^2 < \infty\,, \end{aligned}$$

namely (4) $\Rightarrow$ (3) ((3) $\Rightarrow$ (4) is obvious).

To prove equation (9) I will work out another identity first:

Assertion 1

If A is a trace–class operator on $\mathcal{H}$ and $\{T(g)\}$ is a tomographic set and satisfies (4) (*or* (3)) *then*

$$\mathrm{Tr}\, A = \frac{1}{\tilde{k}} \sum_g T(g)\, A\, T^\dagger(g), \tag{6}$$

with $\tilde{k} \equiv \sum_g \; |\langle\varphi|T(g)|\psi\rangle|^2$ indipendent of the choice of the normalized vectors $|\varphi\rangle, |\psi\rangle \in \mathcal{H}$.

Proof: Hypothesis (3) implies

$$\sum_g \; |\langle a|T(g)|b\rangle \, \langle c|T^\dagger(g)|d\,\rangle| \;\leqslant$$
$$\leqslant \left[\sum_g \; |\langle a|T(g)|b\rangle|^2 \sum_g \; |\langle d|T(g)|c\rangle|^2 \right]^{\frac{1}{2}} < \infty\,, \tag{7}$$

defining unambiguously $\sum_g \; \langle a|T(g)|b\rangle \, \langle c|T^\dagger(g)|d\,\rangle$ for all $|a\rangle$, $|b\rangle$, $|c\rangle$, $|d\,\rangle \in \mathcal{H}$, in accordance with [11]. Since the complete space $\mathcal{H}$, as a consequence of Riesz–Fréchet representation theorem [9], is also weakly complete, we may infer, by virtue of (7), that the sequence of the partial sums $\sum_g^n \; T(g)\, |u\rangle\langle v|\, T^\dagger(g)$ is weakly convergent, as $n \to \infty$, for all $|u\rangle, |v\rangle \in \mathcal{H}$. The operator $I_{uv} \equiv \sum_g T(g)\, |u\rangle\langle v|\, T^\dagger(g)$ can then be defined as the weak limit of such sequence, as $n \to \infty$. Using equation (1) and its adjoint and rearranging the sum we immediately get $D(h)\, I_{uv}\, D^\dagger(h) = I_{uv}$, which, due to Shur's first lemma (for a proof in the infinite–dimensional case refer to [8]), is equivalent to $I_{uv} = k_{uv}\, I$, where $k_{uv} \in \mathbb{C}$ depends on $|u\rangle, |v\rangle$ and I is the identity in $\mathcal{H}$. Analogously, equation (2) entails that $\widetilde{I}_{uv} \equiv \sum_g \; T^\dagger(g)\, |u\rangle\langle v|\, T(g)$ is a multiple of the identity. Given a normalized $|w\rangle \in \mathcal{H}$, k_{ww} may be easily evaluated, in fact $k_{ww} = \langle w|k_{ww}\, I|w\rangle = \langle w|I_{ww}|w\rangle = \sum_g \; |\langle w|T(g)|w\rangle|^2$, where the series could be interchanged with the inner product because of the way I_{uv} is defined. The constant k_{uv} may be expressed in terms of k_{ww} in the following manner (let $|a\rangle, |b\rangle$ be generic vectors): $\langle a|I_{uv}|b\rangle = \langle v|\widetilde{I}_{ba}|u\rangle = \langle v|u\rangle \, \langle w|\widetilde{I}_{ba}|w\rangle = \langle v|u\rangle \, \langle a|I_{ww}|b\rangle$, which means that $I_{uv} = k_{w,w}\, \langle v|u\rangle\, I$. The choice $|u\rangle = |v\rangle = |\psi\rangle$ normalized and the calculation of the mean value of the last equation on the normalized vector $|\varphi\rangle$ produces $\sum_g \; |\langle\varphi|T(g)|\psi\rangle|^2 = k_{w,w}$, proving that $\sum_g \; |\langle\varphi|T(g)|\psi\rangle|^2$ is indipendent of the vectors $|\varphi\rangle, |\psi\rangle$ (for as long as their norm is 1) and will therefore be indicated simply by $\tilde{k}$. Schmidt decomposition of a trace–class operator A, *i.e.* $A = \sum_i a_i\, |u_i\rangle\langle v_i|$, where $\{|u_i\rangle\}$ e $\{|v_i\rangle\}$ are orthonormal sequences and $\sum_i a_i < \infty$, $a_i > 0\ \forall i$, helps showing that $\sum_g \; T(g)\, A\, T^\dagger(g)$ is meaningful. It is indeed sufficient to check the absolute convergence of the expression

$$\sum_g \langle a|T(g)\, A\, T^\dagger(g)|b\rangle =$$
$$= \sum_g \sum_i a_i \, \langle a|T(g)|u_i\rangle \, \langle v_i|T^\dagger(g)|b\rangle, \tag{8}$$

for all $|a\rangle, |b\rangle \in \mathcal{H}$, to insure the validity of the definition of $\sum_g \; T(g)\, A\, T^\dagger(g)$ as the weak limit of $\sum_g^n \; T(g)\, A\, T^\dagger(g)$, as $n \to \infty$. The inequality

$$\sum_g \; |\langle a|T(g)|u_i\rangle \, \langle v_i|T^\dagger(g)|b\rangle| \leqslant$$
$$\leqslant \left[\sum_g \; |\langle a|T(g)|u_i\rangle|^2 \sum_g \; |\langle b|T(g)|v_i\rangle|^2 \right]^{\frac{1}{2}} = \tilde{k},$$

together with $a_i > 0$ and $\sum_i a_i < \infty$, guarantees that the sum of the absolute values of the terms in (8) is $\leqslant \tilde{k} \sum_i \; a_i < \infty$. Because of the absolute convergence we can also rearrange the order of the two sums, obtaining the assertion's thesis:

$$\frac{1}{\tilde{k}} \sum_g \; T(g)\, A\, T^\dagger(g) = \frac{1}{\tilde{k}} \sum_i \sum_g \; a_i\, T(g)|u_i\rangle \, \langle v_i|\, T^\dagger(g) =$$
$$= \sum_i \; a_i \, \langle v_i|u_i\rangle\, I = \mathrm{Tr}\, A\,.$$

Now, finally, equation (9):

Assertion 2

The operator identity

$$A = \frac{1}{\tilde{k}} \sum_g \; \mathrm{Tr}\big[A\, T(g)\big]\; T^\dagger(g) \tag{9}$$

holds, when A, T and $\tilde{k}$ are defined as in assertion 1.

Proof: Let O be an invertible trace–class operator. Using (6) twice, it is straightforward to check that

$$\sum_g \; \mathrm{Tr}[A\, T^\dagger(g)]\, O\, T(g) = \sum_g \; \mathrm{Tr}[\, T(g) O]\, T^\dagger(g) A\,. \tag{10}$$

Expanding the trace on the complete orthonormal sequence $\{|\varphi_i\rangle\}$, with the help of equation (6) again, we may write

$$\frac{1}{\tilde{k}} \sum_g \; \mathrm{Tr}[\, T(g) O] \langle\varphi_i|\, T^\dagger(g) A|\varphi_j\rangle =$$
$$= \sum_h \langle\varphi_h|O\, \mathrm{Tr}\,[|\varphi_h\rangle\langle\varphi_i|]\, A|\varphi_j\rangle = \langle\varphi_i|OA|\varphi_j\rangle\,. \tag{11}$$

Equations (10) and (11) give $\frac{1}{\tilde{k}} \sum_g \mathrm{Tr}[A T^\dagger(g)] O T(g) = OA$, which is equivalent to (9) because of the invertibility of O.

Note [12] is devoted to a brief comment on some technical features of assertions 1 and 2.

Before we start to examine physical cases, I will cast light on some aspects of (9). It is well–known that Hilbert–Schmidt operators form a Hilbert space, usually denoted as $\sigma_c(\mathcal{H})$, with inner product $(A, B)_o \equiv \mathrm{Tr}[A^\dagger, B]$ [13], and that the space $\tau_c(\mathcal{H})$ of trace–class

operators is contained in $\sigma_c(\mathcal{H})$. If we define $P(g) = \tilde{k}^{-\frac{1}{2}}\, T^\dagger(g)$, equation (9) can formally be rewritten as

$$A = \sum_g \,(\, P(g), A\,)_o \, P(g)\,. \tag{12}$$

This equality is of simple interpretation: all vectors of $\tau_c(\mathcal{H})$ can be expanded in terms of a closure relation, resorting to elements $P(g)$ that are not necessarily in $\sigma_c(\mathcal{H})$ but that belong to a larger set, in the same way that occurs with the expansion of a vector in terms of generalized vectors. A comparison will make the situation even clearer. If we identify $\sigma_c(\mathcal{H})$ with $L^2(\mathbb{R})$ (the space of square–integrable functions on $\mathbb{R}$), then $\tau_c(\mathcal{H})$ corresponds to the space $\mathcal{S}(\mathbb{R})$ (test functions on $\mathbb{R}$ decreasing rapidly at infinity) and $\mathcal{B}(\mathcal{H})$ to the space of tempered distributions $\mathcal{S}'(\mathbb{R})$ dual to $\mathcal{S}(\mathbb{R})$. This analogy suggests that formula (12) is also valid for $A \in \sigma_c(\mathcal{H})$ if we define the inner product $(P(g), A)_o$ as the limit of $(P(g), A_n)_o$, with the sequence of trace–class operators $\{A_n\}$ converging to A. Naturally, not all closure relation arise from a group. It is possible to use conditions similar to (1) and (2) defined on sets that are not groups and still obtain (9). Or even more generally, there are cases of spectral decompositions in $\sigma_c(\mathcal{H})$ that do not satisfy anything like (1) and (2) (for example the eigenvectors of a self–adjoint operator from $\sigma_c(\mathcal{H})$ to $\sigma_c(\mathcal{H})$ give a closure relation in $\sigma_c(\mathcal{H})$). However, from an operative point of view, these more general approaches to formula (9) are not very useful. Groups are indeed simple objects to be dealing with and they produce quite a large number of interesting results. Moreover, closure relations that do not exhibit a group structure usually derive from a group. To clarify this point, we must keep in mind that nothing in (12) guarantees that $\{P(g)\}$ is a complete orthonormal system. We know it is complete, because of (12), but we cannot be sure that it is orthonormal with respect to the inner product $(\ ,\ )_o$. And, in fact, generally it is not orthonormal, *i.e.* the set $\{P(g)\}$ is often overcomplete. If an orthonormal system is extracted from the original overcomplete set, the group properties may disappear and we would be left with a spectral decomposition that is not generated by a group, even if it originated in a group. We will encounter an example of this circumstance afterwards.

Equation (6) is a closure relation itself. Choosing $A = |v\rangle\langle v|$, with $|v\rangle$ normalized and arbitrary, it states that $\{P(g)|v\rangle\}$ is a complete set in $\mathcal{H}$. In other terms, the complete set $\{P(g)|v\rangle\}$ in $\mathcal{H}$ corresponds to the complete set $\{P(g)\}$ in $\sigma_c(\mathcal{H})$. Note that this connection is not a consequence of the specific context ($\mathcal{G}$ being a group and T satisfying (1) and (2)) in which we proved formulas (6) and (9).

So far, I assumed that $\mathcal{G}$ was discrete. Nonetheless, (6) and (9) apply to other situations. It is useful to recall that every unitary irriducible representation of a compact group is finite–dimensional (meaning that $\mathcal{H}$ is finite–dimensional), whereas every unitary irriducible representation of a non–compact group is infinite–dimensional (with the exception of the trivial representation). Hence, for a finite group or a compact Lie group, the mathematical problem simplifies: $\dim(\mathcal{H}) < \infty$, $\tau_c(\mathcal{H}) = \sigma_c(\mathcal{H}) = \mathcal{B}(\mathcal{H})$ and the convergence of $\tilde{k}$ is always granted. In particular, for a finite group, by tracing both members of (6), we immediately recognize that $\tilde{k} = \frac{[\mathcal{G}]}{\dim(\mathcal{H})}$, with $[\mathcal{G}]$ indicating the order of $\mathcal{G}$. For a compact Lie group, formulas (6) and (9) are obtained by substituting $\frac{1}{[\mathcal{G}]}\sum_g$, appearing in (6) and (9) for a finite group, with $\int_{\mathcal{G}} \mathrm{d}\mu(g)$, where $\mathrm{d}\mu(g)$ is Haar's invariant measure for $\mathcal{G}$. Similarly, the formal substitution $\sum_g \to \int_{\mathcal{G}} \mathrm{d}\mu(g)$ allows to write (6) and (9) for a non–compact Lie group, with a discrete group as the starting point. A warning is necessary in this case, however. For non–compact groups $\int_{\mathcal{G}} \mathrm{d}\mu(g)$ is not convergent; moreover, differently from the compact case, right and left invariance may correspond to two different $\mathrm{d}\mu(g)$ [14]. Only when they coincide, *i.e.* only when $\mathcal{G}$ is unimodular, formulas (6) and (9) are applicable.

Equation (9) (and (6)) deserves its own self–existence as a pure mathematical result, but my initial goal was different. I was concerned with the physical problem of measuring the density matrix, and, thus far, (9) does not give us any clue on how to solve such problem. I will now show how (9) is, in actuality, much nearer to the solution than what may appear. If $\mathcal{H}$ is the Hilbert space associated with the physical system under consideration, then the density matrix ρ is an element of $\tau_c(\mathcal{H})$ and, consequently, can be written in place of A in formula (9). Moreover, if each $T(g)$ is self–adjoint or is a function of a self–adjoint operator, then we can evaluate the trace in (9) over its complete set of eigenvectors $\{|g,t\rangle\}$. This operation yields an expression containing quantities of the form $\langle g,t|\rho|g,t\rangle$, which can be interpreted as the probability that a measurement of $T(g)$ gives the eigenvalue $t(g)$ corresponding to $|g,t\rangle$ (the case of $t(g)$ degenerate could be treated analogously). These quantities are, in principle, experimentally accessible and will be indicated with $p(g,t)$. Formula (9) then becomes $\rho = \sum_{g,t} p(g,t)[\frac{1}{\tilde{k}} t(g) T^\dagger(g)]$, where if the group is not discrete, or if $T(g)$ has continuous spectrum, sums must be replaced by integrals (obviously there could eventually be both sums and integrals). The observation that the eigenvectors $|g,t\rangle$ and $|g',t\rangle$ are not necessarily different even if $g \neq g'$ suggests to divide $\mathcal{G}$ in classes $\mathcal{G}_i$, requiring the property that all $T(g)$ corresponding to the same class have the same eigenvectors. We would then write $\rho = \sum_{i,t} p(i,t) \sum_{g\in\mathcal{G}_i} [\frac{1}{\tilde{k}} t(g) T^\dagger(g)] \equiv \sum_{i,t} p(i,t) K(i,t)$, where the operator $K(i,t)$ is usually called pattern function (again, if i is a continuous index then $\sum_i \to \int \mathrm{d}\mu(i)$). The formula

$$\rho = \sum_{i,t} p(i,t) K(i,t) \tag{13}$$

is the essence of most state reconstruction methods. It

states that measuring the probabilities $p(i,t)$ and calculating $K(i,t)$ is all that is needed in order to obtain ρ. The peculiarity of quantum tomography is that $K(i,t)$ does not need to be determined by solving an inverse problem, it is explicitly given by $\frac{1}{k}\sum_{g\in\mathcal{G}_i} t(g)T^\dagger(g)$.

It took quite long to work out (13), but its generality will show its strength now that we turn to examples.

I will begin with the spin case, as it is the least complex. The (reduced) spin density matrix of one particle with spin S (integer or half–integer) is defined in a Hilbert space $\mathcal{H}_S$, with $\dim(\mathcal{H}_S) = 2S+1$. The compact group $SU(2)$ is particularly suited for the case of arbitrary S, since there exists an irriducible, unitary representation of $SU(2)$ in every finite–dimensional space. If $SU(2)$ is parametrized with (ϑ,φ,ψ) belonging to $[0,\pi]\times[0,2\pi)\times[0,2\pi]$ and if $\vec{n}$ is defined as $(\cos\varphi\sin\vartheta,\sin\varphi\sin\vartheta,\cos\vartheta)$, then the operators $R(\vec{n},\psi)=\mathrm{e}^{-\mathrm{i}\psi\vec{S}\cdot\vec{n}}$, where $\vec{S}$ is the spin operator [14], constitute an irriducible, unitary representation of $SU(2)$ and, consequently, a tomographic set in $\mathcal{H}_S$. Formula (13) then becomes

$$\rho = \int_\Sigma \mathrm{d}\Omega_{\vec{n}} \sum_{m=-S}^{S} p(\vec{n},m)\, K_S(\vec{n},m)\,, \tag{14}$$

where $\mathrm{d}\Omega_{\vec{n}}$ is the area element of the unitary spherical surface Σ, $K_S(\vec{n},m)$ is the pattern function given by $(2S+1)\int_0^{2\pi}\mathrm{d}\psi\,\frac{\sin^2\frac{\psi}{2}}{4\pi^2}\,\mathrm{e}^{\mathrm{i}\psi(\vec{S}\cdot\vec{n}-m)}$, and $p(\vec{n},m)$ is the probability that m is the result of a measurement of $\vec{S}\cdot\vec{n}$. The calculation of $K_S(\vec{n},m)$, the experimental apparatus needed to measure $p(\vec{n},m)$ and some numerical simulations can be found in [8,15]. Because $\dim(\mathcal{H}_S)<\infty$, it is evident that the tomographic set $\{R(\vec{n},\psi)\}$ is overcomplete. It is then possible to choose a finite number of operators $R(\vec{n},\psi)$ and still obtain a closure relation in $\sigma_c(\mathcal{H}_S)$. As previously mentioned, the operation of extraction of a smaller complete set from the entire set $\{R(\vec{n},\psi)\}$ usually produces a class of operators not corresponding to a group. However, this need not always be the case. The dihedral and tetrahedral subgroups of $SU(2)$ [10] can be represented with a finite number of operators $R(\vec{n},\psi)$, producing tomographic formulas for the cases of $S=\frac{1}{2}$ and $S=1$ (for explicit formulas and some further considerations refer to [8,15]). Unfortunately, there is only a finite number of finite subgroups of $SU(2)$; furthermore, the tomographic set associated with the tetrahedral group is still overcomplete. Therefore, waiving the group structure is a necessity for the obtainment of a complete orthonormal system in $\sigma_c(\mathcal{H}_S)$ for a generic S. This does not mean that we have to completely give up the use of the operators $R(\vec{n},\psi)$. Knowing in advance that $\dim(\sigma_c(\mathcal{H}))=(2S+1)^2$, we can just choose $(2S+1)^2$ linearly independent operators $R(\vec{n},\psi)$, that do not need to form or correspond to a group, and apply Gram–Schmidt orthonormalization procedure: $B_1\equiv\frac{R(\vec{n}_1,\psi_1)}{\|R(\vec{n}_1,\psi_1)\|_o}$, $B_2\equiv\frac{R(\vec{n}_2,\psi_2)-(B_1,R(\vec{n}_2,\psi_2))_o\,B_1}{\|R(\vec{n}_2,\psi_2)-(B_1,R(\vec{n}_2,\psi_2))_o\,B_1\|_o}$, …, with $\|O\|_o\equiv\sqrt{(O,O)_o}$, $\forall\,O\in\sigma_c(\mathcal{H})$. By definition, $\{B_i\}$, $i=1,2,\ldots,(2S+1)^2$, is a basis in $\sigma_c(\mathcal{H})$. Nevertheless also $\{B_i^\dagger\}$ is a basis in $\sigma_c(\mathcal{H})$, therefore every A in $\sigma_c(\mathcal{H})$ can be decomposed as $A=\sum_i(B_i^\dagger,A)_o\;B_i^\dagger=\sum_i\mathrm{Tr}[B_i\,A]\,B_i^\dagger$. Because the operators B_i are linear combinations of the operators $R(\vec{n}_i,\psi_i)$, with a little algebra we get $A=\sum_i\mathrm{Tr}\big[R(\vec{n}_i,\psi_i)\,A\big]\,\mathcal{R}_i$, where the $(2S+1)^2$ operators $\mathcal{R}_i$ are linear combinations of the operators $R^\dagger(\vec{n}_i,\psi_i)$ as a result of the reorganization of the sum on i. One may check that $(\mathcal{R}_i^\dagger,R(\vec{n}_j,\psi_j))_o=\delta_{ij}$ (Kronecker's delta), which means that $\{\mathcal{R}_i\}$ is the dual basis of $\{R(\vec{n}_i,\psi_i)\}$ in $\sigma_c(\mathcal{H})$. Calculating the last trace on the eigenstates $|\vec{n}_i,m\rangle$, associated with the eigenvalue $\mathrm{e}^{-\mathrm{i}\psi_i\,m}$ of the operators $R(\vec{n}_i,\psi_i)$, with $A=\rho$, we attain a finite version of (14):

$$\rho = \sum_{i=1}^{(2S+1)^2}\sum_{m=-S}^{S} p(\vec{n}_i,m)\;\mathcal{K}_S(i,m)\,, \tag{15}$$

with $\mathcal{K}_S(i,m)=\mathrm{e}^{-\mathrm{i}\psi_i\,m}\;\mathcal{R}_i$ (the suffix S in $\mathcal{K}_S(i,m)$ is a remainder of the dependence of the operators $\mathcal{R}_i$ on the dimension of $\mathcal{H}_S$). Equations having the same form of (14) or (15) are quite common in the problem of spin state reconstruction (for example [16–18]). Incidentally, we might also observe that the orthonormalization of $(2S+1)^2$ linearly independent projectors $|\vec{n}_i,S\rangle\langle\vec{n}_i,S|$, $i=1,2,\ldots,(2S+1)^2$, instead of the operators $R(\vec{n}_i,\psi_i)$, leads to the same results obtained by Amiet and Weigert [19].

The Hilbert space $\mathcal{H}_C$ associated with a system of n spins is given by the tensor product of the n single–spin spaces. If we were to write (9) for the elements of $\sigma_c(\mathcal{H}_C)$, we would tempted to choose $\mathcal{G}=SU(2)\times\ldots\times SU(2)$ (n times) and $D(g)=T(g)=\bigotimes_{i=1}^n R(\vec{n}_i,\psi_i)$, with $g\in\mathcal{G}$ ($\times$ and $\bigotimes$ denote respectively the direct product of groups and the tensor product of operators). This choice would actually give a valid closure relation in $\sigma_c(\mathcal{H}_C)$ (equation (9)), but the corresponding probabilities in equation (13) would require measurements on single components of the system, which are not always possible. This difficulty, at least for systems of spins $\frac{1}{2}$, can be overcome with a different approach, as illustrated in [20].

As a model for systems associated with infinite–dimensional Hilbert spaces, we can take the space $\mathcal{H}_O$ of one mode of the electromagnetic field. Although the problem is mathematically identical for other systems (for example, $\mathcal{H}_O$ is isomorphic to the space of a spinless, non–relativistic particle in one dimension), quantum optics gives the unique possibility of measuring the equivalent of linear combinations of position and momentum, namely the so–called quadratures $X_\phi\equiv\frac{1}{2}(a^\dagger\mathrm{e}^{\mathrm{i}\phi}+a\,\mathrm{e}^{-\mathrm{i}\phi})$, with $\phi\in\mathbb{R}$ and a and $a^\dagger$ indicating the annihilation and creation operators respectively [21]. This opportunity can be exploited by choosing the non–compact

Lie group of translations in the complex plane, with elements $\alpha \in \mathbb{C}$, as $\mathcal{G}$. Since the displacement operators $\mathsf{D}(\alpha) = \exp(\alpha\, a^\dagger - \alpha^*\, a)$ form an irriducible, unitary ray representation of $\mathcal{G}$, such that $\int_\mathbb{C} \mathrm{d}^2\alpha\ |\langle 0|\mathsf{D}(\alpha)|0\rangle|^2 = \pi$ ($\mathrm{d}^2\alpha \equiv \mathrm{d}(\mathrm{Re}\,\alpha)\,\mathrm{d}(\mathrm{Im}\,\alpha)$ is the invariant measure of the unimodular group of translations in the plane [10] and $|0\rangle$ is the vacuum state), we can set $D(\alpha) = T(\alpha) = \mathsf{D}(\alpha)$. For the purpose of writing (13), we should, however, express the tomographic set in terms of quadratures. This can be achieved by parameterizing $\mathcal{G}$ with $k \in \mathbb{R}$ and $\phi \in [0, \pi)$, related to α by the equation $\alpha = \frac{\mathrm{i}}{2} k\, \mathrm{e}^{\mathrm{i}\phi}$, since $T(\phi, k) \equiv T(\alpha(\phi, k)) = \mathrm{e}^{\mathrm{i}kX_\phi}$. Equation (13) for this case then reads

$$\rho = \int_0^\pi \mathrm{d}\phi \int_{-\infty}^{+\infty} \mathrm{d}x\, p(\phi, x)\, \mathsf{K}(\phi, x)\,, \tag{16}$$

where $p(\phi, x)$ is the probability that measuring X_ϕ we get x and $\mathsf{K}(\phi, x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \mathrm{d}k\, \frac{|k|}{4}\, \mathrm{e}^{\mathrm{i}k(x - X_\phi)}$ (note that $\tilde{k} = \pi$). Equation (16) is the fundamental formula of homodyne tomography, which I will not be discussing here, because the literature on it is already abundant ([4–6] and references therein).

Homodyne tomography is not the only technique that allows to reconstruct the state ρ of one mode of the electromagnetic field. K. Banaszek and K. Wódkiewicz showed how ρ could be determined by measuring, for every $\alpha \in \mathbb{C}$ and $n \in \mathbb{N}$, the probability $p(\alpha, n)$ that n is the number of photons in the state $\mathsf{D}(\alpha)\rho\,\mathsf{D}^\dagger(\alpha)$ [22]. The same result can be recovered with equation (13), if we maintain the choices $\mathcal{G} = \{$Translations in the complex plane$\}$ and $D(\alpha) = \mathsf{D}(\alpha)$, but we select the tomographic set $\{\mathsf{D}^\dagger(\alpha)\, \mathrm{e}^{\mathrm{i}ya^\dagger a}\mathsf{D}(\alpha)\}$, where y can be any real number that is not a multiple of 2π. It is not hard to check that (1) and (2) are satisfied [23] and that $\int_\mathbb{C} \mathrm{d}^2\alpha |\langle 0|\mathsf{D}^\dagger(\alpha)\, \mathrm{e}^{\mathrm{i}ya^\dagger a}\mathsf{D}(\alpha)|0\rangle|^2 = \frac{\pi}{2(1-\cos y)}$. Then equation (13) is simply

$$\rho = \int_\mathbb{C} \mathrm{d}^2\alpha \sum_{n=0}^{+\infty} p(\alpha, n)\ \mathrm{K}_y(\alpha, n)\,, \tag{17}$$

with $\mathrm{K}_y(\alpha, n) = \frac{2(1-\cos y)}{\pi}\, \mathsf{D}^\dagger(\alpha)\, \mathrm{e}^{\mathrm{i}y(n - a^\dagger a)}\mathsf{D}(\alpha)$.

Differently from (14), neither (16) nor (17) can be simplified by extracting a complete subset from the entire tomographic set, since both $P_1(\alpha) \equiv \pi^{-\frac{1}{2}}\mathsf{D}^\dagger(\alpha)$ and $P_2(\alpha) \equiv (\frac{\pi}{2(1-\cos y)})^{-\frac{1}{2}}\mathsf{D}^\dagger(\alpha)\, \mathrm{e}^{-\mathrm{i}ya^\dagger a}\mathsf{D}(\alpha)$ form orthonormal systems: $(P_1(\alpha), P_1(\alpha'))_o = (P_2(\alpha), P_2(\alpha'))_o = \delta(\mathrm{Re}\,\alpha - \mathrm{Re}\,\alpha')\ \delta(\mathrm{Im}\,\alpha - \mathrm{Im}\,\alpha')$, with δ indicating Dirac's delta.

Some examples have shown how different state reconstruction problems can be treated as particular cases of a general method, which can be summarized in (9) and in the consequent (13). Although the theory developed in this letter is quiete comprehensive, there is still space for further generalizations. One possibility is to assume that the mappings D and T are defined on two distinct groups, changing (1) and (2) appropriately. Apparently, this generalization would produce other interesting tomographic formulas. For example, the state ρ of one optical mode could be determined by measuring only the presence (or the absence) of photons in the state $\mathsf{D}(\alpha)\rho\,\mathsf{D}^\dagger(\alpha)$, for every $\alpha \in \mathbb{C}$.

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[11] In what follows I assume that $\mathcal{G}$ is discrete. This is really not a limitation, and I will discuss later how all results remain valid for other types of group. Also, note that the sum on all elements $g \in \mathcal{G}$ in (3) is well–defined. In fact, since $\mathcal{G}$ is discrete, there exists a $1:1$ mapping F between $\mathcal{G}$ and $\mathbb{N}$, therefore the sum over $\mathcal{G}$ can be written as an ordinary numerical series. The assumption (3), namely the absolute convergence of the series in (3), guarantees that the value of the series does not depend on the choice of the bijective correspondence F.

[12] *Note on the proof of assumption 2.* For the sake of simplicity I avoided the explicit check on sums' convergence or on the possibility of reversing the order of two of them. Nevertheless, these tasks could be accomplished by methods analogous to those used in assertion 1.
Note on the hypothesis. Assertion 1 and 2 could have been proved within different assumptions. For example, we could have given up the requirement of D being a representation in favor of less restrictive hypothesis on D but with less generality on f_{ij} and T. Or, requiring the

invertibility of the operators $T(g)$, we could have used in (1) and (2) any family of numbers that belong to a bounded set in $\mathbb{C}$ instead of the phase factors $\mathrm{e}^{\mathrm{i}\alpha_{gh}}$ and $\mathrm{e}^{\mathrm{i}\beta_{gh}}$. Other modifications could have been introduced as well. I will not be concerned with this point, because the hypothesis enunciated at the beginning seems to be the simplest and most convenient one.

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[22] K. Banaszek, K. Wodkiewicz, Phys. Rev. Lett. **76**, 4344 (1996).

[23] The set $\{D^\dagger(\alpha)\,\mathrm{e}^{\mathrm{i}ya^\dagger a}\,D(\alpha)\}$ is a particular case of a class of tomographic sets. In fact, it would be possible to show that, given an irriducible, unitary ray representation D of $\mathcal{G}$, the operators $D^\dagger(g)\,T_0\,D(g)$ form a tomographic set, whenever $\mathcal{G}$ is Abelian and T_0 is such that $T_0\,D(g) = \mathrm{e}^{\mathrm{i}\eta_g}D(\tilde{f}(g))\,T_0$, for all $g \in \mathcal{G}$, with $\eta : \mathcal{G} \to \mathbb{R}$ and with $\tilde{f} : \mathcal{G} \to \mathcal{G}$ satisfying certain rather involved conditions that will not be investigated here.